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# COMPLETE CLASSES OF GUTS WITH VANISHING ONE-LOOP BETA FUNCTIONS

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## Abstract

By explicit solution of the one-loop finiteness conditions for all dimensionless coupling constants (i. e., gauge coupling constant as well as Yukawa and quartic scalar-boson self-interaction coupling constants), two classes of grand unified theories characterized by renormalization-group beta functions which all vanish at least at the one-loop level are constructed and analyzed with respect to the (suspected) appearance of quadratic divergences, with the result that without exception in all of these models the masses of both vector and scalar bosons receive quadratically divergent one-loop contributions.

# 1 Introduction

Supersymmetry—apart from the important fact of being the only one among the possible symmetries of a nontrivial S-operator [1] which up to now has not been discovered experimentally—attracts continuous interest because of its far-reaching ability of softening the high-energy behaviour of quantum field theories.  $N = 1$  supersymmetric theories satisfying two so-called “finiteness conditions” are finite at least up to two loops [2], even if softly broken [3].  $N = 2$  supersymmetric theories satisfying just a single one-loop finiteness condition prove to be finite at all orders of their perturbation expansion [4], and this once again even in case of being softly broken [5]. The most famous representative of this latter class of finite quantum field theories [6] is the well-known  $N = 4$  super-Yang–Mills theory where the single finiteness condition is automatically satisfied [7].

It is no wonder that soon after the discovery of the supersymmetric finite quantum field theories some in a certain sense “inverse” questions have been put forward [8, 9, 10]: Of what kind are the consequences of the requirement of at least perturbative finiteness for the most general renormalizable quantum field theory? Is, in particular, supersymmetry indeed a necessary prerequisite for finiteness? Do there exist any other, non-supersymmetric finite theories [11]?

In the sequel, some attempts have been undertaken to enlarge the class of finite quantum field theories by building non-supersymmetric finite models. For instance, in Ref. [12] two sets of non-supersymmetric grand unified theories have been proposed which are characterized by the demand of vanishing one-loop renormalization-group  $\beta$  functions for all, in any case dimensionless coupling constants of the theory, and explicit models corresponding to the two or three of the allowed gauge groups with smallest dimension have been given.<sup>1</sup>

The main aim of the actual investigation is to analyze the finiteness conditions resulting from the above requirement of vanishing one-loop beta functions by (as far as manageable) algebraic methods in order to determine the complete classes of the corresponding theories. After a brief recall, in Sect. 2, of the definition of these two classes of models we discuss their finiteness conditions, first, in Sect. 3, for the presumably simpler case and then, in Sect. 4, for the certainly more delicate case, and summarize our conclusions in Sect. 5.

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<sup>1</sup> In view of our findings the models of Ref. [12] do not represent the whole truth.

## 2 Two Models with Vanishing One-Loop Beta Functions

The two models [12]—or, more precisely, two classes of models—with vanishing one-loop contributions to the renormalization-group beta functions of all of their dimensionless coupling constants, which are to be considered in the present investigation, are characterized by three main features:

1. Their gauge group  $\mathcal{G}$  is assumed to be some special unitary group  $\text{SU}(N)$ :  $\mathcal{G} = \text{SU}(N)$ .
2. Their particle content is assumed to involve only particles which transform either according to the fundamental representation or according to the adjoint representation of  $\mathcal{G}$ .
3. Their Lagrangian does not involve any dimensional parameter.

The primary advantage of this very specific and simple choice is that in both of these models all (then necessarily dimensionless) couplings may be expressed solely in terms of

- the generators  $T^a$ , with  $a = 1, 2, \dots, N^2 - 1$ , in the *fundamental* representation of  $\mathcal{G}$ , the normalization of which is fixed by their second-order Dynkin index  $T_f$ , defined by  $T_f \delta_{ab} := \text{Tr}(T^a T^b)$ ;
- the generators  $\frac{1}{i} f_{abc}$  in the *adjoint* representation of  $\mathcal{G}$ , where  $f_{abc}$  denote the (completely antisymmetric) structure constants which define the gauge group under consideration; or
- the completely symmetric constants

$$d_{abc} \equiv \frac{\text{Tr}(\{T^a, T^b\} T^c)}{T_f} \quad .$$

The postulated gauge invariance requires, of course, both models to contain (real) gauge vector bosons  $V_\mu^a$  in the adjoint representation of  $\mathcal{G}$ , which enter in the field strength  $F_{\mu\nu}^a \equiv \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + g f_{abc} V_\mu^b V_\nu^c$  as well as in the covariant derivative  $D_\mu \equiv \partial_\mu - i g V_\mu^a T^a$ .

### 2.1 The general model

Apart from the above-mentioned gauge bosons  $V_\mu^a$ , the particle content of this model consists of

- $m$  sets of Dirac fermions  $\Psi_{(k)}$ ,  $k = 1, 2, \dots, m$ , each of these sets in the adjoint representation of  $\mathcal{G}$ ;
- $m$  sets of Dirac fermions  $\chi_{(k)}$ ,  $k = 1, 2, \dots, m$ , each of these sets in the fundamental representation of  $\mathcal{G}$ ;
- $n$  sets of Dirac fermions  $\zeta_{(k)}$ ,  $k = 1, 2, \dots, n$ , each of these sets in the fundamental representation of  $\mathcal{G}$ ;
- real scalar bosons  $\Phi$  in the adjoint representation of  $\mathcal{G}$ ;
- complex scalar bosons  $\varphi$  in the fundamental representation of  $\mathcal{G}$ .

This general model is defined by the Lagrangian [12]

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + i \sum_{k=1}^m \bar{\Psi}_{(k)}^a (\not{D}_{ab} - h_1 f_{abc} \Phi^c) \Psi_{(k)}^b \\
& + i \sum_{k=1}^m \bar{\chi}_{(k)} (\not{D} - i h_2 T^a \Phi^a) \chi_{(k)} + i \sum_{k=1}^n \bar{\zeta}_{(k)} \not{D} \zeta_{(k)} \\
& + \left( i h_3 \sum_{k=1}^m \bar{\chi}_{(k)} T^a \varphi \Psi_{(k)}^a + \text{H. c.} \right) \\
& + \frac{1}{2} (D_\mu \Phi)^T D^\mu \Phi + (D_\mu \varphi)^\dagger D^\mu \varphi - \frac{\lambda_1}{8} (\Phi^T \Phi)^2 - \frac{\lambda_2}{8} (\Phi^a d_{abc} \Phi^b)^2 \\
& - \frac{\lambda_3}{2} (\Phi^T \Phi) (\varphi^\dagger \varphi) - \frac{\lambda_4}{2} (\Phi^a d_{abc} \Phi^b) (\varphi^\dagger T^c \varphi) - \frac{\lambda_5}{2} (\varphi^\dagger \varphi)^2 \quad . \quad (1)
\end{aligned}$$

The fermions  $\chi_{(k)}$  are discriminated from the fermions  $\zeta_{(k)}$  by the fact that the former also undergo Yukawa interactions whereas the latter do not. In addition to the gauge coupling constant  $g$ , this general model involves three Yukawa coupling constants,  $h_1, h_2, h_3$ , and five scalar-boson self-coupling constants,  $\lambda_1, \lambda_2, \dots, \lambda_5$ . Our aim will be to investigate the consequences of the required vanishing of the one-loop beta functions for these coupling constants.

The finiteness condition which may be satisfied most easily is the one for the one-loop contribution to the renormalization of the gauge coupling constant [12, 13]:

$$21 N - 4 [(2 N + 1) m + n] = 1 \quad . \quad (2)$$

It obviously restricts

- the possible gauge groups  $\text{SU}(N)$  to the values  $N = 4\ell + 1$  for  $\ell = 1, 2, \dots$ , i. e., to the values  $N = 5, 9, 13, \dots$ , and

- the multiplicities  $m$  and  $n$  to three possible combinations: from the positivity of the multiplicity  $n$ , i. e.,  $n \geq 0$ , the multiplicity  $m$  is bounded from above by

$$m \leq \frac{21N - 1}{4(2N + 1)} < 3 \quad \text{for arbitrary } N > 0 \quad ,$$

which restricts  $m$  to the values  $m = 0, 1, 2$ , the corresponding values of  $n$  then being fixed by Eq. (2).

## 2.2 The simplified model

This model is obtained from the more general model described above by completely decoupling the fermions  $\Psi_{(k)}$  and  $\zeta_{(k)}$  as well as the scalar bosons  $\varphi$  from the theory. Accordingly, the non-vector particle content of this model consists of

- $m$  sets of Dirac fermions  $\chi_{(k)}$ ,  $k = 1, 2, \dots, m$ , each of these sets in the fundamental representation of  $\mathcal{G}$ ;
- real scalar bosons  $\Phi$  in the adjoint representation of  $\mathcal{G}$ .

Consequently, this simplified model is defined by the Lagrangian [13]

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + i \sum_{k=1}^m \bar{\chi}_{(k)} (\not{D} - i h T^a \Phi^a) \chi_{(k)} \\ & + \frac{1}{2} (D_\mu \Phi)^T D^\mu \Phi - \frac{\lambda_1}{8} (\Phi^T \Phi)^2 - \frac{\lambda_2}{8} (\Phi^a d_{abc} \Phi^b)^2 \quad . \end{aligned} \quad (3)$$

It now involves only one Yukawa coupling constant,  $h$  (the former  $h_2$ ), and only two scalar-boson self-coupling constants,  $\lambda_1, \lambda_2$ .

Finiteness of the one-loop contribution to the renormalization of the gauge coupling constant, as expressed here by the relation [12, 13]

$$21N = 4m \quad , \quad (4)$$

now restricts the possible gauge groups  $SU(N)$  to the values  $N = 4\ell$  for  $\ell = 1, 2, \dots$ , i. e., to the values  $N = 4, 8, 12, \dots$

According to the finiteness condition (4), for any simplified model with vanishing one-loop gauge-coupling beta function there exists a unique relation between the considered gauge group, i. e., the value of  $N$ , and the allowed multiplicity  $m$ . In the subsequent discussion this relation will be taken into account already from the very beginning by introducing, in place of  $N$  and  $m$ , the “group parameter”

$$\ell = \frac{N}{4} = \frac{m}{21} \quad .$$

### 3 Solutions for the Simplified Model

For the simplified model, the conditions for the one-loop contributions to the renormalization-group beta functions of the Yukawa coupling  $h$  and the quartic scalar-boson self-couplings  $\lambda_i$ ,  $i = 1, 2$ , to vanish [12], expressed in terms of the three real and non-negative variables

$$x \equiv \frac{h^2}{g^2} \geq 0$$

and

$$y_i \equiv \frac{\lambda_i}{g^2} \geq 0 \quad , \quad i = 1, 2 \quad ,$$

read for arbitrary gauge groups  $SU(N)$  with  $N = 4\ell$ ,  $\ell = 1, 2, \dots$ :

$$(184\ell^2 - 3)x^2 - 6(16\ell^2 - 1)x = 0 \quad , \quad (5)$$

$$(16\ell^2 + 7)y_1^2 + 24 - 48\ell y_1 + 84\ell x y_1 + \frac{16\ell^2 - 4}{\ell} \left( y_1 y_2 + \frac{1}{2\ell} y_2^2 \right) - 21x^2 = 0 \quad , \quad (6)$$

$$\frac{16\ell^2 - 15}{\ell} y_2^2 + 12\ell - 48\ell y_2 + 12y_1 y_2 + 84\ell x y_2 - 42\ell x^2 = 0 \quad . \quad (7)$$

Quite obviously, Eq. (5) allows for two and only two solutions for the variable  $x$ , viz.

$$x = 0 \quad (8)$$

or

$$x = \frac{6(16\ell^2 - 1)}{184\ell^2 - 3} \quad . \quad (9)$$

Our first step in the course of determination of the complete set of solutions to the set of equations (5) to (7) given above is the proof of the following statement: *For the second, i. e., non-vanishing, solution (9) of Eq. (5) for the variable  $x$  and for arbitrary values of the group parameter  $\ell = 1, 2, \dots$ , the remaining set of equations (6), (7) does not admit real solutions for both of the variables  $y_1$  and  $y_2$ . In other words: In the case of the simplified model (3), irrespective of the considered gauge group  $SU(N)$  with  $N = 4, 8, \dots$ , there are no solutions at all of the one-loop finiteness conditions for Yukawa interaction and quartic scalar-boson self-couplings, Eqs. (5) to (7), with some non-vanishing Yukawa coupling constant  $h$ , i. e., with  $h \neq 0$ .*

The proof of the above statement is based on the (of course, only from the physical point of view necessary) requirement of reality of our three variables  $x, y_1, y_2$ , in particular, of  $y_1$ . Consider, for any value of the variable  $y_2$ , Eq. (6) as a quadratic equation for  $y_1$  and obtain the variable  $y_1$ , as a function of the so far undetermined variable  $y_2$ , as one of the two roots of this quadratic equation. Then reality of  $y_1$  is guaranteed if a certain inequality of the generic form

$$a(\ell) y_2^2 + b(\ell) y_2 + c(\ell) \leq 0 \quad (10)$$

is satisfied, where, after some amount of straightforward algebra, for the non-vanishing solution (9) of Eq. (5) the three coefficient functions  $a(\ell)$ ,  $b(\ell)$ ,  $c(\ell)$  may be cast into the form

$$\begin{aligned} a(\ell) &= \frac{4\ell^2 - 1}{2\ell^2} (8\ell^2 + 9) \quad , \\ b(\ell) &= 12 \frac{4\ell^2 - 1}{184\ell^2 - 3} (32\ell^2 + 15) \quad , \\ c(\ell) &= 3 \frac{813056\ell^6 + 346496\ell^4 - 4764\ell^2 - 315}{(184\ell^2 - 3)^2} \quad . \end{aligned}$$

Recall that the group parameter  $\ell$  is necessarily larger than or equal to 1,  $\ell \geq 1$ . Within this range of  $\ell$

- the coefficient functions  $a(\ell)$  and  $b(\ell)$  are quite obviously positive for all values of  $\ell$  and,
- similarly, the coefficient function  $c(\ell)$  turns out to be a convex function of this parameter  $\ell$ ,

$$\frac{d^2}{d\ell^2} c(\ell) > 0 \quad \text{for } \ell \geq 1 \quad ,$$

and thus to be, in particular, some strictly monotonic increasing function with increasing  $\ell$  and is therefore bounded from below by its value at  $\ell = 1$ :

$$c(\ell) \geq c(1) = \frac{3463419}{32761} = 105.71 \dots > 0 \quad .$$

Accordingly, all coefficient functions in the inequality (10) are strictly positive,

$$\begin{aligned} a(\ell) &> 0 \quad \forall \ell = 1, 2, \dots \quad , \\ b(\ell) &> 0 \quad \forall \ell = 1, 2, \dots \quad , \\ c(\ell) &> 0 \quad \forall \ell = 1, 2, \dots \quad . \end{aligned}$$

Therefore, because of the (again only from the physical point of view, namely, for reasons of stability of the corresponding quantum field theory, necessary) positivity of the quartic scalar-boson self-couplings, in particular, of the scaled variable  $y_2$ , i. e.,  $y_2 \geq 0$ , the inequality (10) cannot be satisfied. This implies that, for the second, non-vanishing solution for  $x$  as given in Eq. (9), there exists no real solution of the set of equations (5) to (7) for the variable  $y_1$ .

The above findings disprove the claim of the authors of Ref. [12] that “for  $\ell \geq 3$  finite solutions with  $h$  as given by Eq. (9) are also possible.” In contrast to this statement, as has been shown just before, there is *no* chance to find physically acceptable, i. e., real, solutions of the set of equations (5) to (7) with  $h \neq 0$  for any value of  $\ell$ .

In Ref. [13] there has been demonstrated that within any simplified model (of the kind introduced in the preceding section) with vanishing Yukawa coupling constant  $h$  the masses of the scalar bosons receive, already at the one-loop level, quadratically divergent contributions. Therefore, as a by-product of the result stated above, already at this very early stage we may ascertain that all theories among the class of simplified models with vanishing one-loop beta functions, which are to be found as solutions of the finiteness conditions (4) and (5) to (7), will be plagued by quadratic divergences in the renormalization of the scalar-boson masses.

As an important consequence of the above introductory statement, it is sufficient to restrict our discussion to the following reduced set of equations for  $x = 0$ :

$$\begin{aligned} (16\ell^2 + 7)y_1^2 + 24 - 48\ell y_1 + \frac{16\ell^2 - 4}{\ell} \left( y_1 y_2 + \frac{1}{2\ell} y_2^2 \right) \\ = 0 \quad , \end{aligned} \tag{11}$$

$$\frac{16\ell^2 - 15}{\ell} y_2^2 + 12\ell - 48\ell y_2 + 12y_1 y_2 = 0 \quad . \tag{12}$$

Our next step is the derivation of both lower as well as upper bounds on the variables  $y_1$  and  $y_2$ . First of all, both variables  $y_1$  and  $y_2$  must be definitely non-vanishing. Examination of the above reduced set of equations for the two cases  $y_1 = 0$  and  $y_2 = 0$ , respectively, more precisely, of Eq. (11) for  $y_1 = 0$ ,

$$12 + \frac{4\ell^2 - 1}{\ell^2} y_2^2 = 0 \quad ,$$



and of Eq. (12) for  $y_2 = 0$ ,

$$\ell = 0 \quad ,$$

shows that the reality of the variable  $y_2$ , on the one hand, and the requirement  $\ell \neq 0$ , on the other hand, demand

$$y_1 \neq 0 \quad \text{and} \quad y_2 \neq 0 \quad .$$

All of the desired bounds on  $y_1$  and  $y_2$  follow from the existence of a single negative term on the left-hand side of each of Eqs. (11) and (12). Consequently, taking into account the positivity of the coefficients in front of the various terms in these two equations, for any valid solution this single negative term has to counterbalance all positive terms and, in particular, not only the sum of all of them but also the sum of any subset of them. Investigation of the terms at most linear in the scaled quartic scalar-boson self-couplings  $y_1$  or  $y_2$  rather trivially leads, from Eq. (11),

$$1 - 2\ell y_1 < 0 \quad ,$$

to the constraint

$$y_1 > \frac{1}{2\ell}$$

and, from Eq. (12),

$$(1 - 4y_2)\ell < 0 \quad ,$$

to the constraint

$$y_2 > \frac{1}{4} \quad .$$

Similarly, inclusion of the terms proportional to the product  $y_1 y_2$  of the two variables  $y_1$ ,  $y_2$  yields, from Eq. (11),

$$6 - 12\ell y_1 + \frac{4\ell^2 - 1}{\ell} y_1 y_2 < 0 \quad ,$$

the constraint

$$y_2 < \frac{6\ell}{4\ell^2 - 1} \left( 2\ell - \frac{1}{y_1} \right) < \frac{12\ell^2}{4\ell^2 - 1} \leq 4$$

and, from Eq. (12),

$$\ell - 4\ell y_2 + y_1 y_2 < 0 \quad ,$$

the constraint

$$y_1 < \ell \left( 4 - \frac{1}{y_2} \right) < 4\ell \quad .$$

Shuffling together all of the above constraints, the numerical values of the variables  $y_1, y_2$  which eventually provide some solution to the set of equations (11), (12) are unavoidably restricted to the two ranges

$$\frac{1}{2\ell} < y_1 < 4\ell \quad , \quad (13)$$

$$\frac{1}{4} < y_2 < 4 \quad . \quad (14)$$

Note that for the variable  $y_1$  both bounds depend on  $\ell$  whereas for the variable  $y_2$  both bounds do not depend on  $\ell$ . Rather the latter ones hold for arbitrary values of the group parameter  $\ell$ .

By application of our above considerations concerning the reality of the variable  $y_2$  to the case  $x = 0$ , we are able to show that we may expect to obtain solutions of the reduced set of equations (11), (12) only for values of the group parameter  $\ell$  larger than or equal to 2, i. e., for  $\ell \geq 2$ .<sup>2</sup> For the vanishing solution (8) of Eq. (5) the three coefficient functions  $a(\ell)$ ,  $b(\ell)$ ,  $c(\ell)$  in the inequality (10) read

$$\begin{aligned} a(\ell) &= \frac{4\ell^2 - 1}{2\ell^2} (8\ell^2 + 9) \quad , \\ b(\ell) &= 24(4\ell^2 - 1) \quad , \\ c(\ell) &= 42 - 48\ell^2 \quad . \end{aligned}$$

Quite obviously, here only the coefficient functions  $a(\ell)$  and  $b(\ell)$  are positive whereas the coefficient function  $c(\ell)$  is negative for arbitrary  $\ell = 1, 2, \dots$ :

$$\begin{aligned} a(\ell) &> 0 \quad \forall \ell = 1, 2, \dots \quad , \\ b(\ell) &> 0 \quad \forall \ell = 1, 2, \dots \quad , \\ c(\ell) &< 0 \quad \forall \ell = 1, 2, \dots \quad . \end{aligned}$$

However, taking into account the lower bound on  $y_2$  from (14),  $y_2 > \frac{1}{4}$ , it is easy to convince oneself that for  $x = 0$  the inequality (10) can only be satisfied for

$$\ell^2 > \frac{295 + \sqrt{85369}}{368} \simeq 1.6 \quad .$$

Accordingly, for the value  $\ell = 1$  of the group parameter  $\ell$  there exists no solution of the reduced set of equations for  $x = 0$ , Eqs. (11), (12).

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<sup>2</sup> This general statement is in accordance with the findings obtained in Ref. [12] by some numerical approach for the two special cases  $\ell = 1$  and  $\ell = 2$ .

Our final step in the course of determination of the complete set of solutions to the set of equations (5) to (7) is the investigation of Eqs. (11), (12) in the limit of infinitely large gauge groups, i. e., for  $\ell \rightarrow \infty$ . In this limit—which is admittedly hard to interpret from any physical point of view—the remaining set of finiteness conditions to be solved, as represented by Eqs. (11), (12), simplifies to the, in a certain sense, “asymptotic” set of equations

$$2\ell^2 y_1^2 + 3 - 6\ell y_1 + 2\ell y_1 y_2 + y_2^2 = 0 \quad , \quad (15)$$

$$4\ell y_2^2 + 3\ell - 12\ell y_2 + 3y_1 y_2 = 0 \quad . \quad (16)$$

Now, the first term on the left-hand side of Eq. (15) is the only one in the above set of equations which, apart from the a priori unknown dependence of the variable  $y_1$  on  $\ell$ , is proportional to  $\ell^2$ . Hence, the only chance for an eventual balance between the contributions of the various positive and negative terms in Eq. (15) is that  $y_1$  is of the form of some constant,  $k$ , divided by the group parameter  $\ell$ :

$$y_1 = \frac{k}{\ell} \quad .$$

In this case, Eqs. (15), (16) read

$$2k^2 + 3 - 6k + 2ky_2 + y_2^2 = 0 \quad , \quad (17)$$

$$4y_2^2 + 3 - 12y_2 + 3\frac{k}{\ell^2}y_2 = 0 \quad . \quad (18)$$

Note that, according to the right-hand inequality in (14), the variable  $y_2$  is bounded from above by a constant which is independent from the group parameter  $\ell$ :  $y_2 < \text{const.}$  For this reason, the last term on the left-hand side of Eq. (18) does not contribute in the limit  $\ell \rightarrow \infty$ , that is, it may be dropped in the limit  $\ell \rightarrow \infty$ . Accordingly, the final form of the asymptotic set of equations (17), (18) is

$$2k^2 + 3 - 6k + 2ky_2 + y_2^2 = 0 \quad , \quad (19)$$

$$4y_2^2 + 3 - 12y_2 = 0 \quad . \quad (20)$$

Now the second of our asymptotic set of equations, Eq. (20), does no more depend on the variable  $k$  (or  $y_1$ , respectively) and therefore may immediately be solved for  $y_2$ , with the “asymptotic” result

$$y_{2,\infty} = \frac{3 \pm \sqrt{6}}{2} = \begin{cases} 2.7247\dots & , \\ 0.2752\dots & . \end{cases} \quad (21)$$

Of course, both numerical values of this asymptotic solution lie within the two bounds on  $y_2$  given by Eq. (14):  $\frac{1}{4} < y_{2,\infty} < 4$ . Upon insertion of this result for  $y_{2,\infty}$  Eq. (19) becomes, for each of the two solutions for  $y_{2,\infty}$  given in Eq. (21), a quadratic equation for the only up to now unknown variable  $k$ ; the two roots of this quadratic equation may be written down easily. The necessary reality of the constant  $k$ , inherited from the reality of  $y_1$ , however, is granted exclusively for the negative sign in front of the square root in Eq. (21) and hence only for the lower asymptotic solution for  $y_2$ , that is, for  $y_{2,\infty} = \frac{1}{2}(3 - \sqrt{6}) = 0.2752\dots$ . For this numerical value of  $y_{2,\infty}$  the two roots of Eq. (19) are

$$k = \frac{3 + \sqrt{6} \pm \sqrt{3(6\sqrt{6} - 13)}}{4} = \begin{cases} 1.9264\dots & , \\ 0.7983\dots & . \end{cases} \quad (22)$$

In summary, the only asymptotic solutions to the set of equations (5) to (7) for  $\ell \rightarrow \infty$  are given by

$$x = 0 \quad , \quad (23)$$

$$y_{1,\infty} = \frac{3 + \sqrt{6} \pm \sqrt{3(6\sqrt{6} - 13)}}{4\ell} = \frac{1}{\ell} \times \begin{cases} 1.9264\dots & , \\ 0.7983\dots & , \end{cases} \quad (24)$$

$$y_{2,\infty} = \frac{3 - \sqrt{6}}{2} = 0.2752\dots \quad . \quad (25)$$

One encounters no problems at all in verifying the consistency of the above asymptotic solution  $y_{1,\infty}$  for the variable  $y_1$ , Eq. (24), with the ( $\ell$ -dependend) bounds on  $y_1$  from the inequalities (13):  $\frac{1}{2\ell} < y_{1,\infty} < 4\ell$  for both solutions in (24) and for arbitrary  $\ell \geq 1$ .

From Eq. (25) we learn that in the limit  $\ell \rightarrow \infty$  the solutions to the set of equations (11), (12) are degenerate with respect to the variable  $y_2$ . For finite  $\ell$ , however, this degeneracy is removed—for every value of  $\ell$  there exist *two but only two* different solutions for  $y_2$ . In order to see this fact, by expressing the variable  $y_1$  from Eq. (12), which is only linear in  $y_1$ , in terms of  $y_2$  and inserting this expression into Eq. (11), we consider the left-hand side of Eq. (11) as a (quartic) function of  $y_2$  only and calculate the position of the three extrema of this function. The relative signs of the values of this function at these extrema then indicate that it possesses only two real zeros, the two solutions for  $y_2$  just mentioned. For each of these solutions the corresponding value of  $y_1$  may then be computed unambiguously from Eq. (12).

For illustrative purposes, we present in Table 1, for some values of the group parameter  $\ell$ , numerical solutions for the two variables  $y_1$  and  $y_2$ , as computed by some standard numerical method for the solution of coupled sets of equations, as well as the corresponding values of the re-scaled variable  $k \equiv y_1 \ell$ . The observed behaviour of the solutions for large values of  $\ell$  eventually might have been expected already from the preceding discussion:  $y_2$  approaches the single asymptotic value (25) while  $k$  approaches the one or the other of its two possible, constant values (22).

Summarizing the whole set of findings with respect to the possible solutions of the set of equations (5) to (7), the following simple picture emerges: The complete spectrum of solutions to the three, Yukawa and quartic scalar-boson self-interaction, finiteness conditions (5) to (7) of the simplified model (3)

- is characterized by a vanishing Yukawa coupling constant  $h$ , i. e., by

$$h = 0 \quad ,$$

and,

- for every value of the group parameter  $\ell$ , consists of precisely two sequences of solutions for the quartic scalar-boson self-interaction coupling constants  $\lambda_1$ ,  $\lambda_2$  normalized to the square of the gauge coupling constant  $g$ ,  $\lambda_1/g^2$  and  $\lambda_2/g^2$ , which,
- starting at  $\ell = 2$  with the numerical values

$$\frac{\lambda_1}{g^2} = 0.7762 \dots \quad , \quad \frac{\lambda_2}{g^2} = 0.3027 \dots \quad ,$$

and

$$\frac{\lambda_1}{g^2} = 0.4362 \dots \quad , \quad \frac{\lambda_2}{g^2} = 0.2865 \dots \quad ,$$

respectively,

- converge for the group parameter  $\ell$  increasing beyond any limits asymptotically towards the behaviour indicated by Eqs. (24) and (25): for  $\ell \rightarrow \infty$

$$\frac{\lambda_1}{g^2} = \frac{3 + \sqrt{6} \pm \sqrt{3(6\sqrt{6} - 13)}}{4\ell} = \frac{1}{\ell} \times \begin{cases} 1.9264 \dots & , \\ 0.7983 \dots & , \end{cases}$$

$$\frac{\lambda_2}{g^2} = \frac{3 - \sqrt{6}}{2} = 0.2752 \dots \quad .$$

Table 1: Numerical solutions for the two variables  $y_1$  and  $y_2$  of the simplified model as well as the re-scaled, asymptotically constant variable  $k \equiv y_1 \ell$ , all of them for various values of the group parameter  $\ell$ . To the precision aimed at here, the asymptotic region “ $\ell \rightarrow \infty$ ” is reached already for  $\ell \simeq 500$ .

$\ell$	$y_1$	$y_2$	$k \equiv y_1 \ell$
2	0.7762... 0.4362...	0.3027... 0.2865...	1.5524... 0.8724...
3	0.5855... 0.2754...	0.2890... 0.2798...	1.7565... 0.8262...
4	0.4574... 0.2033...	0.2832... 0.2777...	1.8297... 0.8132...
5	0.3728... 0.1615...	0.2804... 0.2768...	1.8641... 0.8076...
6	0.3138... 0.1341...	0.2789... 0.2763...	1.8830... 0.8047...
8	0.2377... 0.1002...	0.2773... 0.2758...	1.9019... 0.8018...
10	$1.9108 \dots \times 10^{-1}$ $0.8005 \dots \times 10^{-1}$	0.2765... 0.2756...	1.9108... 0.8005...
50	$3.8516 \dots \times 10^{-2}$ $1.5968 \dots \times 10^{-2}$	0.2753... 0.2752...	1.9258... 0.7984...
100	$1.9264 \dots \times 10^{-2}$ $0.7984 \dots \times 10^{-2}$	0.2752... 0.2752...	1.9264... 0.7984...
500	$3.8528 \dots \times 10^{-3}$ $1.5966 \dots \times 10^{-3}$	0.2752... 0.2752...	1.9264... 0.7983...
1000	$1.9264 \dots \times 10^{-3}$ $0.7983 \dots \times 10^{-3}$	0.2752... 0.2752...	1.9264... 0.7983...
5000	$3.8528 \dots \times 10^{-4}$ $1.5966 \dots \times 10^{-4}$	0.2752... 0.2752...	1.9264... 0.7983...
10000	$1.9264 \dots \times 10^{-4}$ $0.7983 \dots \times 10^{-4}$	0.2752... 0.2752...	1.9264... 0.7983...

## 4 Solutions for the General Model

For the general model, the conditions for the one-loop contributions to the renormalization-group beta functions of the Yukawa couplings  $h_i$ ,  $i = 1, 2, 3$ , as well as of the quartic scalar-boson self-couplings  $\lambda_i$ ,  $i = 1, 2, \dots, 5$ , to vanish [12], expressed in terms of the eight real and non-negative variables

$$x_i \equiv \frac{h_i^2}{g^2} \geq 0 \quad , \quad i = 1, 2, 3 \quad ,$$

and

$$y_i \equiv \frac{\lambda_i}{g^2} \geq 0 \quad , \quad i = 1, 2, \dots, 5 \quad ,$$

read for arbitrary gauge groups  $SU(N)$ :

$$4N(m+1)x_1^2 - 12Nx_1 + 2mx_1x_2 + x_1x_3 \mp \sqrt{x_1x_2}x_3 = 0 \quad , \quad (26)$$

$$\frac{N^2 + 2mN - 3}{N}x_2^2 - \frac{6(N^2 - 1)}{N}x_2 + 4mNx_1x_2 + \frac{N^2 - 1}{N}x_2x_3 \mp 2N\sqrt{x_1x_2}x_3 = 0 \quad , \quad (27)$$

$$\frac{(4m+1)(N^2 - 1) + N}{2N}x_3^2 - \frac{5N^2 + 1}{N}x_3 + Nx_1x_3 + \frac{N^2 - 1}{2N}x_2x_3 \mp 2N\sqrt{x_1x_2}x_3 = 0 \quad , \quad (28)$$

$$(N^2 + 7)y_1^2 + 24 - 12Ny_1 + \frac{4(N^2 - 4)}{N}y_2\left(y_1 + \frac{2}{N}y_2\right) + 2Ny_3^2 + 8mNx_1y_1 + 4mx_2y_1 - 32mx_1^2 - \frac{4m}{N}x_2^2 = 0 \quad , \quad (29)$$

$$\frac{4(N^2 - 15)}{N}y_2^2 - 12Ny_2 + 3N + 12y_1y_2 + y_4^2 + 8mNx_1y_2 + 4mx_2y_2 - 4mNx_1^2 - 2mx_2^2 = 0 \quad , \quad (30)$$

$$4y_3^2 - \frac{3(3N^2 - 1)}{N}y_3 + 6 + (N^2 + 1)y_1y_3 + \frac{2(N^2 - 4)}{N}y_2y_3 + 2(N+1)y_3y_5 + \frac{2(N^2 - 4)}{N^2}y_4^2 + 4mNx_1y_3 + 2mx_2y_3 + 2m\frac{N^2 - 1}{N}x_3y_3 - 8mx_1x_3 - 4m\frac{N^2 - 1}{N^2}x_2x_3 \pm 4m\sqrt{x_1x_2}x_3 = 0 \quad , \quad (31)$$

$$\begin{aligned}
& \frac{N^2 - 12}{N} y_4^2 - \frac{3(3N^2 - 1)}{N} y_4 + 3N + 2y_1 y_4 + \frac{2(N^2 - 8)}{N} y_2 y_4 \\
& + 2y_4 y_5 + 8y_3 y_4 + 4mN x_1 y_4 + 2m x_2 y_4 + 2m \frac{N^2 - 1}{N} x_3 y_4 \\
& - 4mN x_1 x_3 + \frac{4m}{N} x_2 x_3 = 0 \quad , \tag{32}
\end{aligned}$$

$$\begin{aligned}
& 2(N + 4) y_5^2 - \frac{6(N^2 - 1)}{N} y_5 + (N^2 - 1) y_3^2 + \frac{(N^2 - 4)(N - 1)}{2N^2} y_4^2 \\
& + \frac{3(N - 1)(N^2 + 2N - 2)}{2N^2} + 4m \frac{N^2 - 1}{N} x_3 y_5 \\
& - 2m \frac{(N - 1)(N^2 + N - 1)}{N^2} x_3^2 = 0 \quad . \tag{33}
\end{aligned}$$

The indeterminate sign of the terms involving the square root of the variables  $x_1$  and  $x_2$  in Eqs. (26), (27), (28), and (31) is due to the use of more or less the squares of the Yukawa coupling constants  $h_1$  and  $h_2$  as our basic variables, which forces us to consider both options for this sign in this kind of expression:  $h_i/g = \pm\sqrt{x_i}$  for  $i = 1, 2, 3$ .

Unfortunately, the complexity of the set of equations (26) to (33) prevents or at least discourages a thorough analysis similar to the one that has been performed in the case of the simplified model. Therefore, instead of attempting a discussion in full generality, we first focus our attention to the easier to handle special case of vanishing multiplicity  $m$ , by setting  $m = 0$  in Eqs. (26) to (33).

By closer inspection of the above set of equations (26) to (33), our first observation may be summarized in form of the statement: *In the case of vanishing multiplicity  $m$ , that is, for  $m = 0$ , if there exists any solution to the set of finiteness conditions (26) to (33) at all, at least the six different solutions for the three—“Yukawa-interaction-type”—variables  $x_1, x_2, x_3$  to be listed below appear simultaneously with one and the same set(s) of solutions for the five—“self-interaction-type”—variables  $y_1, y_2, y_3, y_4, y_5$ , regardless of the value of the integer  $N$  which characterizes the gauge group  $SU(N)$ : classifying them with respect to those among the variables  $x_1, x_2, x_3$  which vanish, these solutions read*

1. *for all three variables  $x_1, x_2, x_3$  vanishing*

$$x_1 = x_2 = x_3 = 0 \quad ; \tag{34}$$

2. *for both  $x_2$  and  $x_3$  vanishing*

$$x_1 = 3 \quad , \quad x_2 = x_3 = 0 \quad ; \tag{35}$$



3. for both  $x_1$  and  $x_3$  vanishing

$$x_2 = \frac{6(N^2 - 1)}{N^2 - 3} \quad , \quad x_1 = x_3 = 0 \quad ; \quad (36)$$

4. for both  $x_1$  and  $x_2$  vanishing

$$x_3 = \frac{2(5N^2 + 1)}{N^2 + N - 1} \quad , \quad x_1 = x_2 = 0 \quad ; \quad (37)$$

5. for only  $x_3$  vanishing

$$x_1 = 3 \quad , \quad x_2 = \frac{6(N^2 - 1)}{N^2 - 3} \quad , \quad x_3 = 0 \quad ; \quad (38)$$

and

6. for only  $x_2$  vanishing

$$\begin{aligned} x_1 &= \frac{6N^3 + N^2 - 6N - 1}{N(2N^2 + N - 2)} \quad , \quad x_2 = 0 \quad , \\ x_3 &= \frac{4(2N^2 + 1)}{2N^2 + N - 2} \quad . \end{aligned} \quad (39)$$

This observation follows rather trivially from the fact that in the case  $m = 0$

1. there occurs the complete decoupling of Eqs. (26) to (28), which determine the variables  $x_1, x_2, x_3$ , on the one hand, from Eqs. (29) to (33), which in this case determine the variables  $y_1, y_2, y_3, y_4, y_5$  irrespective of the numerical values of the variables  $x_1, x_2, x_3$ , on the other hand, and
2. the set of equations (26) to (28)—which in any case determines the variables  $x_1, x_2, x_3$ —simplifies to

$$4N x_1^2 - 12N x_1 + x_1 x_3 \mp \sqrt{x_1 x_2} x_3 = 0 \quad , \quad (40)$$

$$\begin{aligned} \frac{N^2 - 3}{N} x_2^2 - \frac{6(N^2 - 1)}{N} x_2 + \frac{N^2 - 1}{N} x_2 x_3 \mp 2N \sqrt{x_1 x_2} x_3 \\ = 0 \quad , \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{N^2 + N - 1}{2N} x_3^2 - \frac{5N^2 + 1}{N} x_3 + N x_1 x_3 + \frac{N^2 - 1}{2N} x_2 x_3 \\ \mp 2N \sqrt{x_1 x_2} x_3 = 0 \quad . \end{aligned} \quad (42)$$

If any two among the three variables  $x_1, x_2, x_3$  vanish, two of the three equations (40) to (42) are identically satisfied whereas the respective third one reduces to a quadratic equation for merely that one among these variables which does not vanish a priori. The two roots of this quadratic equation are then represented either by the vanishing value of the corresponding variable, which corresponds to the solution given in (34), or by the value given in Eqs. (35), (36), or (37), respectively. If, on the other hand, one and only one among the three variables  $x_1, x_2, x_3$  is required to vanish, one of the three equations (40) to (42) is identically satisfied. It is then a simple task to extract the values of the other two, non-vanishing variables from the two remaining equations. For one of the latter requirements, namely, for  $x_1 = 0, x_2 \neq 0, x_3 \neq 0$ , there arises a conflict with the necessary positivity of our variables  $x_1, x_2, x_3$  whence the corresponding solution does not exist at all.

From the above discussion it should have become rather clear that, in order to decide whether or not there will be solutions to the set of finiteness conditions (26) to (33) for vanishing multiplicity  $m$  at all, we have to investigate, for  $m = 0$ , the set of equations which under these circumstances determine the variables  $y_1, y_2, y_3, y_4, y_5$  for their own, namely, Eqs. (29) to (33). In the case  $m = 0$  these equations undergo a tremendous simplification, leaving us with the set of equations

$$(N^2 + 7) y_1^2 + 24 - 12 N y_1 + \frac{4(N^2 - 4)}{N} y_2 \left( y_1 + \frac{2}{N} y_2 \right) + 2 N y_3^2 = 0 \quad , \quad (43)$$

$$\frac{4(N^2 - 15)}{N} y_2^2 - 12 N y_2 + 3 N + 12 y_1 y_2 + y_4^2 = 0 \quad , \quad (44)$$

$$4 y_3^2 - \frac{3(3N^2 - 1)}{N} y_3 + 6 + (N^2 + 1) y_1 y_3 + \frac{2(N^2 - 4)}{N} y_2 y_3 + 2(N + 1) y_3 y_5 + \frac{2(N^2 - 4)}{N^2} y_4^2 = 0 \quad , \quad (45)$$

$$\frac{N^2 - 12}{N} y_4^2 - \frac{3(3N^2 - 1)}{N} y_4 + 3 N + 2 y_1 y_4 + \frac{2(N^2 - 8)}{N} y_2 y_4 + 2 y_4 y_5 + 8 y_3 y_4 = 0 \quad , \quad (46)$$

$$2(N + 4) y_5^2 - \frac{6(N^2 - 1)}{N} y_5 + (N^2 - 1) y_3^2 + \frac{(N^2 - 4)(N - 1)}{2 N^2} y_4^2 + \frac{3(N - 1)(N^2 + 2N - 2)}{2 N^2} = 0 \quad . \quad (47)$$

Here, we observe

- that—in contrast to the most general case as represented by the set of equations (29) to (33)—now the left-hand sides of each of the above relations involve, for  $N \geq 5$ , just a single negative term, which has to compensate for all of the positive terms, and
- that each of these single negative terms is linear in just one of the variables  $y_1, y_2, y_3, y_4, y_5$ , each of these variables appearing just once in one of these negative terms.

This observation has two immediate consequences:

1. All of our five self-interaction-type variables  $y_1, y_2, \dots, y_5$  must be definitely non-vanishing:

$$y_1 \neq 0 \quad , \quad y_2 \neq 0 \quad , \quad y_3 \neq 0 \quad , \quad y_4 \neq 0 \quad , \quad y_5 \neq 0 \quad .$$

If anyone of these variables would vanish, obviously at least one among Eqs. (43) to (47) could not be satisfied, namely, that one the left-hand side of which contains the negative term involving this vanishing variable.

2. The obvious necessity of counterbalancing all positive terms on the left-hand sides of Eqs. (43) to (47) by the single negative term imposes a lot of bounds on the variables  $y_1, y_2, \dots, y_5$ . For  $N \geq 5$  the collection of the best of these bounds reads

$$0 < \frac{2}{N} < y_1 < \frac{3(3N^2 - 1)}{N(N^2 + 1)} < \frac{9}{N} \leq \frac{9}{5} \quad , \quad (48)$$

$$\frac{1}{4} < y_2 < \frac{3N^2}{N^2 - 4} \leq \frac{25}{7} \quad , \quad (49)$$

$$0 < \frac{2N}{3N^2 - 1} < y_3 < \frac{3(3N^2 - 1)}{8N} < \frac{9N}{8} \quad , \quad (50)$$

$$\frac{1}{3} < \frac{N^2}{3N^2 - 1} < y_4 < \frac{3(3N^2 - 1)}{N^2 - 12} \leq \frac{222}{13} \quad , \quad (51)$$

$$\frac{1}{4} < \frac{N^2 + 2N - 2}{4N(N + 1)} < y_5 < \frac{3(N^2 - 1)}{N(N + 4)} < 3 \quad . \quad (52)$$

The above bounds may serve to provide some useful guide in any numerical search for solutions of the set of equations (43) to (47).

The smallest possible gauge group which may serve to illustrate the above observation is  $SU(9)$ : here, for  $N = 9$  and  $m = 0$ , the following six solutions for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$ , viz.,

$$x_1 = x_2 = x_3 = 0 \quad ,$$

$$x_1 = 3 \quad , \quad x_2 = x_3 = 0 \quad ,$$

$$x_2 = \frac{80}{13} = 6.1538\dots \quad , \quad x_1 = x_3 = 0 \quad ,$$

$$x_3 = \frac{812}{89} = 9.1235\dots \quad , \quad x_1 = x_2 = 0 \quad ,$$

$$x_1 = 3 \quad , \quad x_2 = \frac{80}{13} = 6.1538\dots \quad , \quad x_3 = 0 \quad ,$$

$$x_1 = \frac{4400}{1521} = 2.8928\dots \quad , \quad x_2 = 0 \quad , \quad x_3 = \frac{652}{169} = 3.8579\dots \quad ,$$

accompany both of the two sets of only numerically found solutions for the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  listed in Table 2. Of course, both of these sets of numerical solutions respect the bounds on the variables  $y_1, y_2, \dots, y_5$  given in Eqs. (48) to (52).

Table 2: Numerical solutions for the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  of the general model with gauge group  $SU(9)$  and multiplicity  $m = 0$ .

Variable	Solution I	Solution II
$y_1$	0.4017...	0.5054...
$y_2$	0.2864...	0.2907...
$y_3$	0.1862...	0.3676...
$y_4$	0.3860...	0.4008...
$y_5$	0.4167...	0.7812...

The mere existence of solutions to the finiteness conditions (26) to (33) for  $N = 9$  and  $m = 0$  clearly contradicts and therefore corrects the impression one might get from Ref. [12] that solutions to this set of equations with  $h_1 = h_2 = h_3 = 0$  do only exist for  $N \geq 13$  but not for  $N = 9$ . While we find indeed no solutions to the set of equations (43) to (47)—which describes the situation realized for  $h_1 = h_2 = h_3 = 0$ —for  $N = 5$ , Table 2 presents the corresponding sets of solutions for  $N = 9$ .

We would like to address the question of the eventual presence of quadratic divergences in the scalar-boson masses also for the class of general models. The relevant analytic expressions for the quadratically divergent one-loop contributions to the masses of both types of scalar bosons in the general model,  $\Phi$  and  $\varphi$ , have been calculated already in Ref. [13]. Apart from some trivial factors, these quadratic divergences turn out to be proportional to some quantities  $Q$  involving the various coupling constants in the theory. Expressed in terms of our variables  $x_i$ ,  $i = 1, 2, 3$ , and  $y_i$ ,  $i = 1, 2, \dots, 5$ , the quantities  $Q$  read in the sector of the scalar bosons  $\Phi$  [13]

$$\begin{aligned} \frac{1}{g^2} Q^{(\Phi)} &= 6N - 4m(2Nx_1 + x_2) \\ &+ (N^2 + 1)y_1 + 2\frac{N^2 - 4}{N}y_2 + 2Ny_3 \end{aligned} \quad (53)$$

and in the sector of the scalar bosons  $\varphi$  [13]

$$\begin{aligned} \frac{1}{g^2} Q^{(\varphi)} &= 3\frac{N^2 - 1}{N} - 4m\frac{N^2 - 1}{N}x_3 \\ &+ (N^2 - 1)y_3 + 2(N + 1)y_5 \quad . \end{aligned} \quad (54)$$

For vanishing multiplicity  $m$  all of the various contributions on the right-hand sides of Eqs. (53) and (54) are strictly positive. This implies that in this case both of the quantities  $Q^{(\Phi)}$  and  $Q^{(\varphi)}$  are necessarily non-vanishing. Hence, we are forced to conclude that for  $m = 0$  one will encounter quadratic divergences in the course of renormalization of scalar-boson masses in every single one among the class of general models, irrespective of the precise numerical values of the involved coupling constants  $h_i$ ,  $i = 1, 2, 3$ , and  $\lambda_i$ ,  $i = 1, 2, 3, 5$ .

For non-vanishing multiplicity  $m$  it is hard to make any general statements concerning the spectrum of solutions to the set of finiteness conditions (26) to (33) based on purely algebraic investigations. It is, however, a rather simple task to determine the corresponding solutions by some numerical method. Here, we merely intend to exemplify this particular state of affairs by reporting in Tables 3 to 6, for both of the two non-vanishing multiplicities  $m$  allowed according to Subsection 2.1 by one-loop finiteness of the gauge coupling constant renormalization,  $m = 1$  and  $m = 2$ , all the sets of solutions obtained numerically within the two smallest possible gauge groups, that is,  $SU(5)$  and  $SU(9)$ , for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$  as well as for the

Table 3: Numerical solutions for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$  and the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  of the general model with gauge group  $SU(5)$  and multiplicity  $m = 1$ .

Variable	Solution I	Solution II
$x_1$	1.4642...	1.4211...
$x_2$	0	1.6806...
$x_3$	1.4303...	2.3612...
$y_1$	0.7059...	0.6594...
$y_2$	1.3872...	1.2933...
$y_3$	0.1692...	0.3235...
$y_4$	1.6480...	1.6765...
$y_5$	0.6067...	1.0385...
$Q^{(\Phi)}$	3.129...	-2.321...
$Q^{(\varphi)}$	-1.719...	-10.708...

Table 4: Numerical solutions for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$  and the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  of the general model with gauge group  $SU(5)$  and multiplicity  $m = 2$ .

Variable	Solution I	Solution II
$x_1$	0.9847...	0.9678...
$x_2$	0	0.3725...
$x_3$	0.9174...	1.1525...
$y_1$	0.6627...	0.6357...
$y_2$	0.7087...	0.6606...
$y_3$	0.1709...	0.1668...
$y_4$	0.9075...	1.0056...
$y_5$	0.4079...	0.5618...
$Q^{(\Phi)}$	-23.882...	-26.663...
$Q^{(\varphi)}$	-11.833...	-19.112...

five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$ . Some but not all of these solutions have already been found in Ref. [12]. In fact, in Ref. [12] exactly one set of solutions per gauge group and multiplicity has been given.

Table 5: Numerical solutions for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$  and the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  of the general model with gauge group SU(9) and multiplicity  $m = 1$ .

Variable	Solution I	Solution II	Solution III	Solution IV
$x_1$	0	0	1.4805...	1.4546...
$x_2$	0	0	0	1.7417...
$x_3$	0	0	1.3988...	2.3294...
$y_1$	0.4017...	0.5054...	0.4336...	0.4149...
$y_2$	0.2864...	0.2907...	1.2385...	1.1947...
$y_3$	0.1862...	0.3676...	0.0923...	0.1756...
$y_4$	0.3860...	0.4008...	1.4849...	1.7329...
$y_5$	0.4167...	0.7812...	0.7222...	1.1369...
$Q^{(\Phi)}$	95.201...	107.041...	5.816...	-0.074...
$Q^{(\varphi)}$	49.901...	71.706...	-1.241...	-19.369...

Table 6: Numerical solutions for the three Yukawa-interaction-type variables  $x_1, x_2, x_3$  and the five self-interaction-type variables  $y_1, y_2, y_3, y_4, y_5$  of the general model with gauge group SU(9) and multiplicity  $m = 2$ .

Variable	Solution I	Solution II	Solution III	Solution IV
$x_1$	0	0	0.9917...	0.9817...
$x_2$	0	0	0	0.3878...
$x_3$	0	0	0.8934...	1.1273...
$y_1$	0.4017...	0.5054...	0.3775...	0.3685...
$y_2$	0.2864...	0.2907...	0.7145...	0.6880...
$y_3$	0.1862...	0.3676...	0.0901...	0.0899...
$y_4$	0.3860...	0.4008...	0.8547...	0.9858...
$y_5$	0.4167...	0.7812...	0.4621...	0.6088...
$Q^{(\Phi)}$	95.201...	107.041...	-44.001...	-46.862...
$Q^{(\varphi)}$	49.901...	71.706...	-20.412...	-34.131...

In the case of vanishing Yukawa-interaction-type variables  $x_1, x_2, x_3$  the set of equations (26) to (33) becomes completely independent from the multiplicity  $m$ : in this case Eqs. (26) to (28) are satisfied identically whereas Eqs. (29) to (33) reduce to the simplified set of equations (43)

to (47). Consequently, for some given gauge group  $SU(N)$ , that is, for a definite choice of the value of  $N$ , one and the same sets of solutions with  $x_1 = x_2 = x_3 = 0$  must appear for all conceivable multiplicities  $m = 0, 1, 2$ . And indeed, by comparing the two solutions given in Table 2 with Solution I and II in Table 5 as well as in Table 6, respectively, the attentive reader will find out, at least for the instance of the gauge group  $SU(9)$ , that this duplication of those solutions where all Yukawa coupling constants vanish, i. e., of the solutions with  $h_1 = h_2 = h_3 = 0$ , really takes place.

Let us bring up the question of quadratic divergences for the very last time. Computing the quantities  $Q^{(\Phi)}$  and  $Q^{(\varphi)}$  from Eqs. (53) and (54) for all sets of numerical solutions for the Yukawa-interaction and scalar-boson self-interaction coupling constants listed in Tables 3 to 6, we find the corresponding values quoted also in these tables. Since all of these values are definitely non-vanishing, we end up with the insight that in each of these examples for general one-loop finite models with non-vanishing multiplicity  $m$  there will arise quadratic divergences for the masses of the scalar bosons.

## 5 Summary and Conclusions

The present discussion has been devoted to the explicit construction of two particular classes—or, more precisely, sequences—of grand unified theories singled out from the most general case by the requirement of vanishing one-loop contributions to all the renormalization-group beta functions of the dimensionless coupling constants in the theory. The particle content of both of these two sequences of models is restricted to transform according to some reducible representation of the gauge group which is (equivalent to) the direct sum of (certain multiples of) the fundamental and the adjoint representations of the gauge group.

For both types of models, that is, the more or less rather simple one as well as the slightly more sophisticated one, we intended to extract in a totally algebraic way from the one-loop finiteness conditions for the Yukawa and scalar-boson self-interaction coupling constants as much information as possible about the spectrum of theories fulfilling these requirements. In the case of the class of simplified models, because of, on the one hand, the rather high degree of simplicity of this class of models and, on the other hand, the existence, as a consequence of the finiteness of the renormalization of the gauge coupling constant at the



one-loop level, of a unique relationship between the gauge group and the otherwise arbitrary multiplicity of the fermionic particle content, we were able to draw a coherent picture. Herein we clearly discern some sort of “ladder with converging uprights,” which is formed by precisely two distinct solutions for each valid gauge group. In the evidently more complicated case of the class of general models we found it advisable to distinguish between the conceivable values of the relevant multiplicity of fermion representations contained in the above-mentioned reducible representation and, in particular, to investigate theories with vanishing or either of the two admissible non-vanishing multiplicities separately. For vanishing multiplicity we worked out, in a very similar manner as in the case of the class of simplified models, the most essential features of this subset of models. For non-vanishing multiplicity we simply gave the explicit sets of grand unified theories which correspond to the two smallest possible and therefore perhaps most interesting gauge groups.

Since all the models in the above two classes of theories have been subjected just to the requirement of vanishing one-loop beta functions, we have been interested, of course, in the question whether or not there arise quadratic divergences. Taking advantage from the fact that, for both classes of theories, the explicit expressions for the quadratically divergent one-loop contributions to all the scalar-boson masses may be read off immediately from Ref. [13], we are led to the conclusion that the scalar bosons of all theories within the complete class of simplified models as well as both types of scalar bosons within the few explicitly constructed examples of models of the more general kind receive, not very surprisingly, already at the one-loop level quadratically divergent contributions to their masses. Furthermore, as far as the masses of the gauge vector bosons are concerned, it has been demonstrated already in Ref. [13] that, within both classes of theories, for the vector-boson masses the same statement as above on the occurrence of quadratic divergences holds in any case.

Needless to say that all our statements based on one or the other algebraic argument may be and have been checked by a corresponding solution of the set of equations under consideration by some numerical methods.

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